Journal of Applied Mechanics and Technical Physics, Vol. 50, No. 6, pp. 936–943, 2009

PULSATING MOTION OF AN INHOMOGENEOUS SOLID SPHERE IN A VIBRATING LIQUID

V. L. Sennitskii

UDC 532.582

The problem of rotational vibrations of an inhomogeneous solid body (a sphere) in a uniformly vibrating ideal liquid under gravity is considered. New hydromechanical effects are reported.

Key words: *liquid vibrations, vibration of an inhomogeneous solid.*

Problems of motion of an homogeneous solid in uniformly and nonuniformly vibrating ideal liquids were solved in [1–9].

1. The present papers studies the following problem. An inhomogeneous absolutely solid (a sphere) is present in an ideal incompressible liquid which has no outer boundary (Fig. 1). There is a constant gravity field. At the initial time $t = 0$, the liquid and the sphere are at rest relative to the inertial rectangular coordinate system x, y, z; the center of the sphere is in the plane $x = 0$. At the subsequent times $t > 0$, the hydromechanical system is subjected to external actions; as a result, the liquid at infinity vibrates along the z axis in a specified manner with period $T_{\rm liq}$; the body vibrates around its axis located on the x axis; the liquid current is potential. It is required to determine the stable equilibrium positions and motion of the body.

Let $A > 0$, Γ, and $m > 0$ be the radius, boundary, and mass of the sphere, respectively; $\boldsymbol{n}$ is the outward unit normal to Γ, $\boldsymbol{S}_{\rm c} \neq 0$ is the radius-vector of the center of the sphere, $\boldsymbol{S}_{\rm i} + \boldsymbol{S}'_{\rm i}$ is the radius-vector of the center of inertia of the sphere (the vector $\boldsymbol{S}_{\rm i}$ is different from zero and is in the plane $x = 0$; the vector $\boldsymbol{S}'_{\rm i}$ is equal to zero or parallel to the x axis), $\boldsymbol{e}_x = \{1, 0, 0\}$, $\boldsymbol{e}_z = \{0, 0, 1\}$, $\boldsymbol{g} = -g\boldsymbol{e}_z$ $(g > 0)$ is the acceleration due to gravity, α is the angle between the vectors $\boldsymbol{g}$ and $\boldsymbol{S}_{\rm i}$, $\alpha_{\rm ini}$ is the value of α at $t = 0$, $\boldsymbol{\Omega} = (d\alpha/dt)\boldsymbol{e}_x$ is the angular velocity of rotation of the sphere, $0 \leqslant \beta \leqslant \pi$ is the angle between the vectors $\boldsymbol{S}_{\rm i}$ and $\boldsymbol{S}_{\rm c}$, L is the angular momentum of the sphere with respect to the x axis, $M_{\rm liq}$ is the moment of the forces exerted by the liquid on the sphere with respect to the x axis, $M_{\rm g}$ is the moment of the gravity forces acting on the sphere with respect to the x axis, f is a function of t, ρ and Φ are the density and velocity potential of the liquid, respectively, P is the pressure in the liquid, and $\boldsymbol{U} = U\boldsymbol{e}_z = \hat{U}u\boldsymbol{e}_z$ is the liquid velocity at infinity ($\hat{U}$ is the largest value of $|\boldsymbol{U}|$, and $u = {\rm Re}\sum\limits_{k=1}^{\infty} u_k\, {\rm e}^{2k\pi i t/T_{\rm liq}}$, where u_k are constants and ${\rm Re}\sum\limits_{k=1}^{\infty} u_k = 0$).

Position of the body is set by a angle α [any positions of the body characterized by values of α differing by $2l\pi$ $(l = \pm 1, \pm 2, \ldots)$ are identical].

The equation for the angular momentum of the body and the moments of the forces acting on it (with respect to the x axis), Cauchy–Lagrange integral, the equation of continuity and the conditions that should be satisfied on the boundary of the body, at infinity, and at the initial time are written as

$$\frac{dL}{dt} = M_{\rm liq} + M_{\rm g}; \tag{1.1}$$

$$\frac{\partial \Phi}{\partial t} + \frac{1}{2}\,(\nabla\Phi)^2 + \frac{P}{\rho} + gz = f; \tag{1.2}$$

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; igil@hydro.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 50, No. 6, pp. 27–35, November–December, 2009. Original article submitted January 21, 2009.

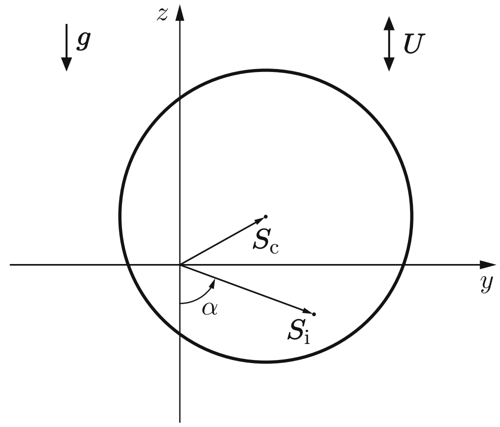

Fig. 1. Hydromechanical system (liquid and solid) and the coordinate system in the plane $x = 0$.

$$\Delta\Phi = 0; \tag{1.3}$$

$$\boldsymbol{n} \cdot \nabla\Phi = \boldsymbol{n} \cdot (\boldsymbol{\Omega} \times \boldsymbol{S}_\mathrm{c}) \qquad \text{in} \quad \Gamma; \tag{1.4}$$

$$\nabla\Phi \to \boldsymbol{U} \qquad \text{at} \quad x^2 + y^2 + z^2 \to \infty; \tag{1.5}$$

$$\alpha = \alpha_\mathrm{ini}, \qquad \frac{d\alpha}{dt} = 0 \qquad \text{at} \quad t = 0. \tag{1.6}$$

2. According to (1.3)–(1.5),

$$\Phi = U[z + B\cos(\alpha+\beta)] + \frac{A^3\{U[z + B\cos(\alpha+\beta)] - B\,(d\alpha/dt)[y\cos(\alpha+\beta) + z\sin(\alpha+\beta)]\}}{2\{x^2 + [y - B\sin(\alpha+\beta)]^2 + [z + B\cos(\alpha+\beta)]^2\}^{3/2}}, \tag{2.1}$$

where $B = |\boldsymbol{S}_\mathrm{c}|$.

Using (1.1), (1.2), and (2.1), we find

$$\frac{d^2\alpha}{d\tau^2} = -\varepsilon\lambda\Big\{\varepsilon[(\varkappa - \cos\beta)\sin\alpha - \sin\beta\cos\alpha] - \frac{3}{2}\,\mu\,\frac{du}{d\tau}\,\sin(\alpha+\beta)\Big\}. \tag{2.2}$$

Here

$$\tau = \frac{t}{T_\mathrm{liq}}, \quad \varepsilon = \sqrt{\frac{g}{B}}\,T_\mathrm{liq}, \quad \varkappa = \frac{mC}{(4\pi/3)A^3B\rho}, \quad \lambda = \frac{2}{1 + I/((2\pi/3)A^3B^2\rho)}, \quad \mu = \frac{\hat{U}}{\sqrt{Bg}}$$

($C = |\boldsymbol{S}_\mathrm{i}|$ and I is the moment of inertia of the body with respect to the x axis).

Problem (1.6), (2.2) can be considered to be the main problem of motion of the body.

3. Let the liquid at infinity be at rest. Then, the following equation holds:

$$\frac{d^2\alpha}{dt^2} = -\lambda\,\frac{g}{B}\,[(\varkappa - \cos\beta)\sin\alpha - \sin\beta\cos\alpha]. \tag{3.1}$$

Let us consider problem (1.6), (3.1).

If the relations $\beta = 0$ and $\varkappa = 1$ hold simultaneously, the body is in the state of indifferent equilibrium for any α_ini. From now on, we assume that the above relations do not hold simultaneously unless otherwise specified.

We rearrange Eq. (3.1) as

$$\frac{d^2\alpha}{dt^2} = -\omega_0^2\sin(\alpha - \chi_0). \tag{3.2}$$

Here

$$\omega_0 = \sqrt{\lambda\,\frac{g}{B}\,\sqrt{\varkappa^2 - 2\varkappa\cos\beta + 1}}; \tag{3.3}$$

$$\chi_0 = \arccos \frac{\varkappa - \cos\beta}{\sqrt{\varkappa^2 - 2\varkappa\cos\beta + 1}}. \tag{3.4}$$

According to (1.6) and (3.2), the body can be in the state of equilibrium. The positions of equilibrium of the body are determined by the following values of α_{eq} and the angle α:

$$\alpha_{\text{eq}} = \chi_0; \tag{3.5}$$

$$\alpha_{\text{eq}} = \chi_0 + \pi. \tag{3.6}$$

The equilibrium of the body in position (3.5) is stable, and at position (3.6), it is unstable. At $\alpha_{\text{ini}} \neq \alpha_{\text{eq}}$, the body performs vibrations about the position (3.5) at cyclic frequency (3.3). Problem (1.6), (3.2) has the solution

$$t = \Bigl(N - \frac{1}{2}\Bigr)\frac{T_0}{2} + \frac{(-1)^N}{\omega_0} F\Bigl(\arcsin \frac{\sin((\alpha - \chi_0)/2)}{\sin((\alpha_{\text{ini}} - \chi_0)/2)}, \sin\frac{\alpha_{\text{ini}} - \chi_0}{2}\Bigr)$$
$$(\chi_0 < \alpha_{\text{ini}} < \chi_0 + \pi), \tag{3.7}$$

where

$$T_0 = \frac{4}{\omega_0} F\Bigl(\frac{\pi}{2}, \sin\frac{\alpha_{\text{ini}} - \chi_0}{2}\Bigr) \tag{3.8}$$

is the period of vibrations of the body, F is an elliptic integral of the first kind, and $N = 1, 2, \ldots$ is the number of the half-period of vibrations of the body within which t changes:

$$(N-1)T_0/2 \leqslant t \leqslant NT_0/2.$$

From (3.7) and (3.8), in particular, it follows that, for small vibrations of the body for values of $\alpha_{\text{ini}} - \chi_0$ that are small compared to unity (in a linear approximation with respect to $\alpha_{\text{ini}} - \chi_0$),

$$\alpha = \chi_0 + (\alpha_{\text{ini}} - \chi_0)\cos(2\pi t/T_\Omega). \tag{3.9}$$

Here

$$T_\Omega = 2\pi/\omega_0 \tag{3.10}$$

is the period of small vibrations of the body.

4. We address problem (1.6), (2.2) and consider values of ε that are small compared to unity.

4.1. We use the averaging method [10]. Let

$$s = \int_0^\tau u\, d\tau$$

and η be a variable (angle) that satisfies the relation

$$\alpha = \eta + (3/2)\varepsilon\lambda\mu s \sin(\eta + \beta); \tag{4.1}$$

$$\xi = \frac{1}{\varepsilon}\frac{d\eta}{d\tau}. \tag{4.2}$$

Using (4.1) and (4.2), we bring (2.2) to a normal system of equations. Expanding the right side of the equation containing $d\xi/d\tau$ in powers of ε, and retaining only the main term of the expansion, we transform from the normal system of equations to the system of equations in nonstandard form

$$\frac{d\eta}{d\tau} = \varepsilon\xi,$$
$$\frac{d\xi}{d\tau} = -\varepsilon\lambda\Bigl\{(\varkappa - \cos\beta)\sin\eta - \sin\beta\cos\eta - \frac{9}{8}\lambda\mu^2 s\frac{du}{d\tau}\sin[2(\eta+\beta)] + 3u\xi\cos(\eta+\beta)\Bigr\}. \tag{4.3}$$

We average (4.3) with respect to the explicitly contained variable τ. The resulting system of equations reduces to the equation

$$\frac{d^2\eta}{d\tau^2} = -\varepsilon^2\lambda\{(\varkappa - \cos\beta)\sin\eta - \sin\beta\cos\eta + \nu\sin[2(\eta+\beta)]\}, \tag{4.4}$$

where

$$\nu = \frac{9}{8}\,\lambda\mu^2 \int\limits_{\tau}^{\tau+1} u^2\,d\tau = \frac{9}{16}\,\lambda\mu^2 \sum_{k=1}^{\infty} |u_k|^2.$$

Using (1.6) and (4.1), we obtain

$$\eta = \alpha_{\mathrm{ini}}, \qquad \frac{d\eta}{d\tau} = 0 \qquad \text{at} \quad \tau = 0. \tag{4.5}$$

4.2. We consider problem (4.4), (4.5) for $\beta = 0$, $\beta = \pi$.

4.2.1. According to (4.4), for $\beta = 0$ the following equation holds:

$$\frac{d^2\eta}{d\tau^2} = -\varepsilon^2\lambda(\varkappa - 1 + 2\nu\cos\eta)\sin\eta. \tag{4.6}$$

Let

$$0 < \nu \leqslant |\varkappa - 1|/2.$$

According to (4.5) and (4.6), the positions of equilibrium of the body are determined by the values η_{eq} of the angle η:

$$\eta_{\mathrm{eq}} = 0; \tag{4.7}$$

$$\eta_{\mathrm{eq}} = \pi. \tag{4.8}$$

If $\varkappa - 1 > 0$, the equilibrium of the body in position (4.7) is stable, and in position (4.8), it is unstable. The body vibrates around the position (4.7) at the cyclic frequency

$$\omega = \sqrt{\lambda(\varkappa - 1 + 2\nu)}\;\varepsilon/T_{\mathrm{liq}}. \tag{4.9}$$

If $\varkappa - 1 < 0$, the equilibrium of the sphere in position (4.8) is stable, and in position (4.7), it is unstable. The body vibrates around position (4.8) at a cyclic frequency

$$\omega = \sqrt{\lambda(-\varkappa + 1 + 2\nu)}\;\varepsilon/T_{\mathrm{liq}}. \tag{4.10}$$

Let

$$\nu > |\varkappa - 1|/2. \tag{4.11}$$

According to (4.5) and (4.6), the positions of equilibrium of the body are determined by values (4.7) and (4.8) and the following values η_{eq} of the angle η:

$$\eta_{\mathrm{eq}} = \arccos\frac{-\varkappa + 1}{2\nu}, \qquad \eta_{\mathrm{eq}} = -\arccos\frac{-\varkappa + 1}{2\nu}. \tag{4.12}$$

The equilibrium of the body in each of positions (4.7) and (4.8) is stable, and in positions (4.12), it is unstable.

4.2.2. According to (4.4), for $\beta = \pi$ the following equation holds:

$$\frac{d^2\eta}{d\tau^2} = -\varepsilon^2\lambda(\varkappa + 1 + 2\nu\cos\eta)\sin\eta. \tag{4.13}$$

Let

$$0 < \nu \leqslant (\varkappa + 1)/2.$$

According to (4.5) and (4.13), the positions of equilibrium of the body are determined by values (4.7) and (4.8) of the angle η. The equilibrium of the body in position (4.7) is stable, and in position (4.8), it is unstable. The body vibrates around position (4.7) at the cyclic frequency

$$\omega = \sqrt{\lambda(\varkappa + 1 + 2\nu)}\;\varepsilon/T_{\mathrm{liq}}. \tag{4.14}$$

Let

$$\nu > (\varkappa + 1)/2. \tag{4.15}$$

According to (4.5) and (4.13), the positions of equilibrium of the body are determined by values (4.7) and (4.8) and the following values η_{eq} of the angle η:

$$\eta_{\rm eq} = -\arccos\frac{\varkappa+1}{2\nu} + \pi, \qquad \eta_{\rm eq} = \arccos\frac{\varkappa+1}{2\nu} - \pi. \tag{4.16}$$

The equilibrium of the body in each of positions (4.7) and (4.8) is stable, and in positions (4.16), it is unstable.

4.2.3. The results presented in Secs. 4.2.1 and 4.2.2 suggest that the following effect takes place.

The vibrations of the liquid at infinity [for $\beta = 0$ if (4.11) is satisfied and for $\beta = \pi$ if (4.15) is satisfied] cause an increase in the number of positions of equilibrium of the body and, in particular, the number of positions of its stable equilibrium. For

$$\beta = 0, \quad -\arccos\frac{-\varkappa+1}{2\nu} < \alpha_{\rm ini} < 0, \quad \text{and} \quad 0 < \alpha_{\rm ini} < \arccos\frac{-\varkappa+1}{2\nu},$$

the body vibrates around position (4.7) at a cyclic frequency (4.9). For

$$\beta = 0, \quad \arccos\frac{-\varkappa+1}{2\nu} < \alpha_{\rm ini} < \pi, \quad \text{and} \quad \pi < \alpha_{\rm ini} < -\arccos\frac{-\varkappa+1}{2\nu} + 2\pi,$$

the body vibrates around position (4.8) at the cyclic frequency (4.10). Vibrations of the body around position (4.7) (for $\beta = 0$) are ordinary if $\varkappa - 1 > 0$, and they are paradoxical if $\varkappa - 1 < 0$. Vibrations of the body around position (4.8) (for $\beta = 0$) are ordinary if $\varkappa - 1 < 0$, and they are paradoxical if $\varkappa - 1 > 0$. For

$$\beta = \pi, \quad \arccos\frac{\varkappa+1}{2\nu} - \pi < \alpha_{\rm ini} < 0, \quad \text{and} \quad 0 < \alpha_{\rm ini} < -\arccos\frac{\varkappa+1}{2\nu} + \pi,$$

the body vibrates around position (4.7) at the cyclic frequency (4.14). For

$$\beta = \pi, \quad -\arccos\frac{\varkappa+1}{2\nu} + \pi < \alpha_{\rm ini} < \pi, \quad \text{and} \quad \pi < \alpha_{\rm ini} < \arccos\frac{\varkappa+1}{2\nu} + \pi,$$

the body vibrates around position (4.8) at the cyclic frequency

$$\omega = \sqrt{\lambda(-\varkappa - 1 + 2\nu)}\, \varepsilon/T_{\rm liq}. \tag{4.17}$$

Vibrations of the body around position (4.7) (for $\beta = \pi$) are ordinary. Vibrations of the body around position (4.8) (for $\beta = \pi$) are paradoxical.

We note that, according to (4.9), (4.10), (4.14), and (4.17), the cyclic frequencies of paradoxical vibrations of the body are lower than the cyclic frequencies of its ordinary vibrations. Moreover, the cyclic frequencies of paradoxical vibrations of the body can be arbitrarily low.

A characteristic feature of paradoxical vibrations of the body is that they are inverted; i.e., the body performs them while inverted.

There is an effect known as the Kapitza pendulum [11] the effect of paradoxical inverted vibrations of a solid in the absence of a liquid (under vibrations of the suspension axis of the body). The results of investigation of the problem (4.4), (4.5) for $\beta = 0$ and $\beta = \pi$, including (4.6)–(4.10), (4.12)–(4.14), (4.16), and (4.17), in particular, indicate the existence of an effect similar to the Kapitza pendulum in hydromechanics.

4.3. We consider problem (4.4), (4.5) for $\beta \neq 0$ and $\beta \neq \pi$.

4.3.1. Let us determine whether a stable equilibrium position of the body exists for both $\nu = 0$ and $\nu > 0$ (for $\beta \neq 0$, $\beta \neq \pi$).

According to (3.5), the existence of an equilibrium position of the body for $\nu > 0$ that coincides with its stable equilibrium position for $\nu = 0$ is equivalent to satisfying the condition

$$\eta_{\rm eq} = \chi_0. \tag{4.18}$$

Using (3.4), (4.4), (4.5), and (4.18), we obtain

$$\chi_0 = -\beta + \pi/2, \tag{4.19}$$

where

$$0 < \beta = \arccos(1/\varkappa) < \pi/2 \quad (\varkappa > 1)$$

is a solution of the equation

$$\varkappa \cos\beta - 1 = 0. \tag{4.20}$$

The equilibrium of the body in position (4.18), (4.19) is stable if

$$0 < \nu \leqslant \sqrt{\varkappa^2 - 1}/2, \tag{4.21}$$

and it is unstable if

$$\nu > \sqrt{\varkappa^2 - 1}/2. \tag{4.22}$$

The body can vibrate around position (4.18), (4.19); the period of small vibrations of the body (T_{small}) is expressed by the formulas

$$T_{\text{small}} = \frac{2\pi T_{\text{liq}}}{\sqrt{\lambda(\sqrt{\varkappa^2 - 1} - 2\nu)\,\varepsilon}} \quad \text{at} \quad 0 < \nu < \frac{\sqrt{\varkappa^2 - 1}}{2}; \tag{4.23}$$

$$T_{\text{small}} = \frac{4F(\pi/2, 1/\sqrt{2}\,)\,T_{\text{liq}}}{\sqrt{\lambda\nu}\,(\alpha_{\text{ini}} + \beta - \pi/2)\varepsilon} \quad \text{at} \quad \nu = \frac{\sqrt{\varkappa^2 - 1}}{2}.$$

We note that Eqs. (3.3), (3.10), (4.20), and (4.23) imply that

$$T_{\text{small}} \to T_\Omega \quad \text{at} \quad \nu \to 0.$$

The results obtained in this subsection indicate that if (4.19) is satisfied, then, according to (3.5), (4.18), (4.21), and (4.22), there is a position of the body that is its stable equilibrium position for

$$0 \leqslant \nu \leqslant \sqrt{\varkappa^2 - 1}/2.$$

4.3.2. Let

$$Q = \sqrt{[\sin\beta - \nu\sin(\eta + 2\beta)]^2 + [\varkappa - \cos\beta + \nu\cos(\eta + 2\beta)]^2} > 0.$$

We rearrange Eq. (4.4) as

$$\frac{d^2\eta}{d\tau^2} = -\varepsilon^2\lambda Q\sin(\eta - \chi), \tag{4.24}$$

where $\chi = \chi(\eta)$ is the variable (angle) that satisfies the relations

$$\sin\chi = [\sin\beta - \nu\sin(\eta + 2\beta)]/Q; \tag{4.25}$$

$$\cos\chi = [\varkappa - \cos\beta + \nu\cos(\eta + 2\beta)]/Q \tag{4.26}$$

and the condition

$$\chi = \chi_0 \quad \text{at} \quad \nu = 0. \tag{4.27}$$

According to (4.5) and (4.24), the positions of equilibrium of the body are determined by the values η_{eq} and η'_{eq} of the angle η which are solutions of the equations

$$\eta_{\text{eq}} = \chi(\eta_{\text{eq}}); \tag{4.28}$$

$$\eta'_{\text{eq}} = \chi(\eta'_{\text{eq}}) + \pi. \tag{4.29}$$

The equilibrium of the body in each of positions (4.28) and (4.29) can be both stable and unstable. In particular, the equilibrium of the body is stable in position (4.28) if

$$1 - \frac{d\chi}{d\eta}\Big|_{\eta=\eta_{\text{eq}}} > 0, \tag{4.30}$$

and it is unstable in position (4.29) if

$$1 - \frac{d\chi}{d\eta}\Big|_{\eta=\eta'_{\text{eq}}} > 0. \tag{4.31}$$

4.3.3. We consider formulas (4.25), (4.26), and (4.28)–(4.31) for values of ν that are small compared to unity.

According to (4.25)–(4.27), (4.30), and (4.31) for $\nu \to 0$, the equilibrium of the body in position (4.28) is stable, and in position (4.29), it is unstable.

Let us examine to what extent the stable equilibrium positions of the body differ for $\nu = 0$ and $\nu > 0$ (in an approximation linear in ν). Using (3.4) and (4.25)–(4.28), we obtain

$$\Delta\chi = \eta_{\text{eq}} - \chi_0 = 2\nu \frac{\varkappa(1 - \varkappa\cos\beta)\sin\beta}{(\varkappa^2 - 2\varkappa\cos\beta + 1)^{3/2}}. \tag{4.32}$$

Formula (4.32) establishes that each of the relations $\Delta\chi > 0$, $\Delta\chi < 0$ or

$$\Delta\chi = 0, \tag{4.33}$$

can hold and (according to Sec. 4.3.1), relation (4.33) holds if (4.20) holds.

We note that relations (3.4) and (4.32) imply that

— if $0 < \chi_0 < \pi/2$, then $\Delta\chi > 0$, $\Delta\chi < 0$, and $\Delta\chi = 0$;

— if $\pi/2 \leqslant \chi_0 < \pi$, then $\Delta\chi > 0$.

The results obtained in this subsection, in particular, indicate that liquid vibrations can cause a shift in the stable equilibrium position of the body both toward its inverted position and from its inverted position.

4.4. We examine problem (4.4), (4.5) for values of ν larger compared to unity.

4.4.1. According to (4.4), as $\nu \to \infty$, the following equation holds:

$$\frac{d^2\eta}{d\tau^2} = -\varepsilon^2\lambda\nu\sin[2(\eta + \beta)]. \tag{4.34}$$

According to (4.5) and (4.34), the positions of equilibrium of the body are determined by the following values η_{eq} of the angle η:

$$\eta_{\text{eq}} = -\beta; \tag{4.35}$$

$$\eta_{\text{eq}} = -\beta + \pi/2, \qquad \eta_{\text{eq}} = -\beta - \pi/2; \tag{4.36}$$

$$\eta_{\text{eq}} = -\beta + \pi. \tag{4.37}$$

The equilibrium of the body in each of positions (4.35) and (4.37) is stable, and in positions (4.36), it is unstable. For

$$-\beta - \pi/2 < \alpha_{\text{ini}} < -\beta \qquad \text{and} \qquad -\beta < \alpha_{\text{ini}} < -\beta + \pi/2,$$

the body vibrates around position (4.35). For

$$-\beta + \pi/2 < \alpha_{\text{ini}} < -\beta + \pi \qquad \text{and} \qquad -\beta + \pi < \alpha_{\text{ini}} < -\beta + 3\pi/2,$$

the body vibrates around position (4.37). In the approximation of small vibrations of the body, problem (4.5), (4.34) has solutions

$$\eta = -\beta + (\alpha_{\text{ini}} + \beta)\cos(\omega_\infty t); \tag{4.38}$$

$$\eta = -\beta + \pi + (\alpha_{\text{ini}} + \beta - \pi)\cos(\omega_\infty t), \tag{4.39}$$

where

$$\omega_\infty = \sqrt{2\lambda\nu}\,\varepsilon/T_{\text{liq}}. \tag{4.40}$$

4.4.2. Use (3.3), (3.9), and (4.38)–(4.40) leads to the conclusion that the cyclic frequency (4.40) of vibrations of the body around positions (4.35) and (4.37) coincides with the cyclic frequency (3.3) of vibrations of the body around position (3.5) in an effective gravity field whose intensity $\boldsymbol{g}'$ satisfies the condition

$$|\boldsymbol{g}'| = 2\nu g/\sqrt{\varkappa^2 - 2\varkappa\cos\beta + 1};$$

the direction $\boldsymbol{g}'$ can be chosen such that the position of equilibrium of the body that corresponds to (3.5) coincides with position (4.35) or with position (4.37).

4.5. We compare problem (4.4), (4.5) and problem (4.5), (4.34). Equation (4.4) for $\beta = 0$, $\varkappa = 1$, and $\nu > 0$ becomes Eq. (4.34) for $\beta = 0$:

$$\frac{d^2\eta}{d\tau^2} = -\varepsilon^2\lambda\nu\sin 2\eta. \tag{4.41}$$

This means that, if $\beta = 0$, the results for $\nu \to \infty$ obtained in Sec. 4.4.1 are applicable for $\varkappa = 1$ and $\nu > 0$. In particular, for any $\nu > 0$, if relations $\beta = 0$, $\varkappa = 1$ are satisfied simultaneously, the body has the stable equilibrium positions

$$\eta_{\rm eq} = 0, \qquad \eta_{\rm eq} = \pi,$$

about which the body vibrates at the cyclic frequency (4.40) [which also directly follows from problem (4.5), (4.41)].

Thus, due to vibrations of the liquid, the body can leave the state of indifferent equilibrium, take stable equilibrium positions, and perform vibrations whose period is different from the period with which the liquid at infinity vibrates.

5. The study led to the detection of new hydromechanical effects, of which the effect of the exit of the body from the state of indifferent equilibrium deserves special attention. This effect is of obvious independent interest; at the same time, it illustrates with striking clarity the difference between the "usual" and "unusual," which is a manifestation of changes in the properties of the hydromechanical system due to external actions.

The results of this work can be used, in particular, to solve the problem of controlling vibrational processes in hydromechanical systems.

REFERENCES

1. V. L. Sennitskii, "Motion of a circular cylinder in a vibrating liquid," *J. Appl. Mech. Tech. Phys.*, No. 5, 630–623 (1985).
2. V. L. Sennitskii, "Motion of a sphere in a fluid caused by vibrations of another sphere," *J. Appl. Mech. Tech, Phys.*, No. 4, 501–505 (1986).
3. B. A. Lugovtsov and V. L. Sennitskii "Motion of a body in a vibrating liquid," *Dokl. Akad. Nauk SSSR*, **289**, No. 2, 314–317 (1986).
4. V. L. Sennitskii, "Motion of inclusions in uniformly and nonuniformly vibrating liquid," in: *Proc. of the Int. Workshop on G-Jitter* (Potsdam, USA, June 13–19, 1993), Clarkson Univ., Potsdam (1993), pp. 178–186.
5. V. L. Sennitskii, "Motion of a sphere in a vibrating liquid in the presence of a wall," *J. Appl. Mech. Tech. Phys.*, **40**, No. 4, 662–668 (1999).
6. I. E. Kareva and V. L. Sennitskii "Motion of a circular cylinder in a vibrating liquid," *J. Appl. Mech. Tech. Phys.*, **42**, No. 2, 276–278 (2001).
7. O. S. Pyatigorskaya and V. L. Sennitskii, "Motion of a solid body a nonuniformly vibrating liquid," *Vestnik. Novosib. Gos. Univ., Ser. Mat., Mekh., Informatika*, **2**, No. 2, 55–59 (2002).
8. O. S. Pyatigorskaya and V. L. Sennitskii, "Motion of a sphere in a liquid caused by vibrations of another sphere," *J. Appl. Mech. Tech. Phys.*, **45**, No. 4, 542–545 (2004).
9. V. L. Sennitskii, "Motion of inclusion in uniformly and nonuniformly vibrating liquids," *J. Appl. Mech. Tech. Phys.*, **48**, No. 1, 65–70 (2007).
10. N. N. Bogolyubov and Yu. A. Mitropol'skii, *Asymptotic Methods in the Theory of Nonlinear Vibrations* [in Russian], Nauka, Moscow (1974).
11. P. L. Kapitza, "Pendulum with a vibrating suspension," *Usp. Fiz. Nauk*, **44**, No. 1, 7–20 (1951).